Drive-train (Drag-Race) Model

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A robot drive-train acceleration model was developed for Team 1640 late in 2008 as a tool for drive-train design. The approach and mathematics used in developing this model were documented only in manuscript at that time. A slightly revised and improved edition of the model math is documented here.

Scope

The model determines acceleration, velocity and position versus time of a robot which is at rest at $t = 0$, and where full motor power is applied for $t \geq 0$. A 5 s duration is modeled (10 s under Lunacy conditions).

Wheel slippage is accounted for.

All weight-bearing wheels are assumed to be driven (important).

Robot mass, gear reduction ratio, wheel diameter, static & kinetic friction coefficients (wheel to floor), drive-train torque loss and number of CIM motors are variable.

CIM motors are default. Substituting other drive motors for CIMs is straightforward, but there are no provisions made for mixed motor systems.

Wheel torque assumes 4 wheels and equal weight distribution. These assumptions are unimportant to the model outcome.

Definitions

t = time (s)
x = position (ft)
v = velocity (ft/s)
a = acceleration (ft/s^2)
m = robot mass (lb_m)
r_w = wheel radius (ft)
n = number of drive motors
G = overall gear reduction ratio (motor to wheel)
μ_s = static coefficient of friction
μ_k = kinetic coefficient of friction
ν_M = motor rotational speed (revolutions/s)
ν_{MU} = unloaded motor rotational speed (88.5 revolutions/s for CIM)
ν_W = wheel rotational speed (revolutions/s)
τ_M = motor torque for one motor (ft lb_f)
τ_{MS} = motor torque at stall (1.789 ft lb_f for CIM)

τ_{ML} = motor torque loss due to transmission losses (0.223 ft lb_f used)
τ_W = wheel torque for all drive wheels combined (ft lb_f)
I = motor current @ 12 VDC (amps)
I_U = motor current, unloaded @ 12 VDC (2.7 amps for CIM)
I_S = motor current, stalled @ 12 VDC (133 amps for CIM)
g_c = conversion factor (32.174 lb_m ft/lb_f s^2)
F_{fs} = maximum static frictive force (lb_f)
F_{fk} = maximum kinetic frictive force (lb_f)
F_d = maximum drive force under non-slip conditions (lb_f)
wheel slip = logical – TRUE or FALSE
slip Δv = wheel slip velocity (ft/s)
Δt = time step used for numerical solution

Initial & Boundary Conditions

At t = 0 s:

x = 0 ft
v = 0 ft/s

At $t \geq 0$ s:

Motor at full power (follow motor curves)

The Mathematics – without wheel slippage

The relationship between acceleration, velocity and position is:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \qquad \text{eq 1}$$

An object accelerates due to applied force.

$$F = \frac{ma}{g_c} = \frac{m}{g_c}\frac{dv}{dt} \qquad \text{eq 2}$$

therefore

$$\frac{dv}{dt} = \frac{F}{m} g_c \qquad \text{eq 3}$$

Note: If we were using a consistent unit system, such as SI or cgs (a.k.a. metric), the g_c factor would be unnecessary.

Ignoring wheel slippage (for now), force is applied via drive wheels (the number of drive wheels is not important).

$$F_d = \frac{\tau_W}{r_W} \qquad \text{eq 4}$$

Note that τ_W is the total torque available to all drive wheels combined.

Torque at wheels depends upon motor torque, gearing, drive-train losses and number of motors. Earlier versions of this model used a drive-train efficiency factor (ε), so that $\tau_W = \varepsilon G n \tau_M$. This approach has the drawback of reducing drive-train losses to zero as speed increases and the motors unload. Clearly unrealistic as the maximum robot speed was then based on unloaded motor speeds. I don't know that a constant torque loss is accurate, but it's better than a constant efficiency.

$$\tau_W = Gn(\tau_M - \tau_{ML}) \qquad \text{eq 5}$$

so that:

$$F_d = \frac{Gn(\tau_M - \tau_{ML})}{r_W} \qquad \text{eq 6}$$

A motor running at full power (12 VDC) will have a rotational speed depending upon the resistance (torque) applied to that motor. The relationship between torque and speed is linear, so that:

$$\nu_M = \nu_{MU} - \frac{\nu_{MU}}{\tau_{MS}}\tau_M = \nu_{MU}\left(1 - \frac{\tau_M}{\tau_{MS}}\right) \qquad \text{eq 7}$$

or

$$\tau_M = \tau_{MS}\left(1 - \frac{\nu_M}{\nu_{MU}}\right) \qquad \text{eq 8}$$

Motor speed relates to wheel speed:

$$\nu_W = \frac{\nu_M}{G} \qquad \text{eq 9}$$

and (with no wheel slippage) to velocity:

$$v = 2\pi r_W \nu_W = \frac{2\pi r_W \nu_M}{G} \qquad \text{eq 10}$$

or

$$\nu_M = \frac{vG}{2\pi r_W} \qquad \text{eq 11}$$

Substituting eq 11 into eq 8

$$\tau_M = \tau_{MS}\left(1 - \frac{vG}{2\pi r_W v_{MU}}\right) \qquad \text{eq 12}$$

eq 12 into eq 6

$$F_d = \frac{Gn\left[\tau_{MS}\left(1 - \frac{vG}{2\pi r_W v_{MU}}\right) - \tau_{ML}\right]}{r_W} \qquad \text{eq 13}$$

eq 13 into eq 3

$$\frac{dv}{dt} = \frac{Gng_c\left[\tau_{MS}\left(1 - \frac{vG}{2\pi r_W v_{MU}}\right) - \tau_{ML}\right]}{mr_W} = \frac{Gn\tau_{MS}g_c}{mr_W}\left(1 - \frac{\tau_{ML}}{\tau_{MS}} - \frac{G}{2\pi r_W v_{MU}}v\right) \qquad \text{eq 14}$$

eq 14 is a 1^{st}-order ordinary differential equation. It may be solved for v(t) analytically via straightforward integration and use of initial conditions (@ t = 0; v = 0) to solve the constant of integration. A second integration (a bit uglier & less straightforward, but quite do-able) can solve for x(t), again utilizing initial conditions (@ t = 0; x = 0) to solve the second constant of integration.

This analytical solution, while elegant, is only valid for the case that the wheels don't slip. Since this is a real poor assumption, we need to use an alternative, numerical approach to solve this differential equation.

Wheel Slippage

Maximum frictive force is

$$F_f = \mu F_n \qquad \text{eq 15}$$

where:

- μ = the coefficient of friction; and
- F_n = the normal force holding the two frictive surfaces together

For a robot on a flat playing surface, F_n is the robot's weight (not mass – weight is a force). A robot having a mass of 120 lb_m will apply a weight or force (F_n) of 120 lb_f onto a flat playing surface. eq 2 still applies here ($F = ma/g_c$), but on Earth, a = g = 32.174 ft/s^2. Since g_c = 32.174 lb_m ft/lb_f s^2, mass and weight are numerically identical for English units on Earth. Metric units neatly avoid this confusion.

The maximum frictive force is different between objects at rest and moving objects. For objects at rest (not sliding relative to each other), the maximum frictive force is:

$$F_{fs} = \mu_s F_n \qquad \text{eq 16}$$

For objects sliding relative to each other:

$$F_{fk} = \mu_k F_n \qquad \text{eq 17}$$

μ_s (the static coefficient of friction) is larger than μ_k (the kinetic coefficient of friction). It takes less force to keep something sliding than to start it sliding.

So, for our drag race model, if wheels are not already slipping, they will not slip unless $F_d > F_{fs}$.

Once wheels are slipping, they will not stop slipping until $F_d < F_{fk}$.

The numerical calculations need to test for wheel slippage based on the above criteria.

During the period of time when the wheels are slipping, the accelerating force applied to the robot is F_{fk}, not F_d.

So, as long as wheels are slipping, eq 14 needs to be replaced by

$$\frac{dv}{dt} = \frac{g_c}{m} F_{fk} = \frac{g_c}{m} \mu_k F_n = \frac{g_c}{m} \mu_k m \frac{g}{g_c} = \mu_k g \qquad \text{eq 18}$$

When the wheels slip, wheel torque is determined by kinetic frictional force.

$$\tau_W = F_{fk} r_W = \frac{g}{g_c} \mu_k m r_W \qquad \text{eq 19}$$

likewise motor torque.

$$\tau_M = \frac{\tau_W}{Gn} + \tau_{ML} \qquad \text{eq 20}$$

and motor & wheels speeds from eq 7 & eq 9.

<u>Numerical Solution</u>

Differential equations are solved via numerical methods by breaking the problem up into small segments along the independent variable (time, in this case). *dv/dt* is first solved at t = 0 using the initial condition @ t = 0; v = 0. Other useful values are calculated as well. A small time step is made (typically 0.01 s is good for this model) and a new velocity is estimated

$$v_{i+1} = v_i + \Delta t \frac{dv}{dt}_i \quad \text{eq 21}$$

Position is estimated using the average velocity during the time step

$$x_{i+1} = x_i + \Delta t \frac{v_i + v_{i+1}}{2} \quad \text{eq 22}$$

The analytical solution to eq 14 is useful here is verifying the accuracy of the results and in selecting an appropriate Δt value. The Δt of 0.01 s was selected via this approach (forcing the model to ignore wheel slippage).

At each time step, an assessment needs to be made as to whether the wheels are slipping. The logical variable *wheel slip,* calculated for each time step, is used to monitor wheel slippage. The logic for setting *wheel slip* is:

- If wheel $slip_{(i-1)}$ = FALSE then
    - if $F_d \le F_{fs}$ then wheel $slip_{(i)}$ = FALSE
    - if $F_d > F_{fs}$ then wheel $slip_{(i)}$ = TRUE
- If wheel $slip_{(i-1)}$ = TRUE then
    - if $F_d \le F_{fk}$ then wheel $slip_{(i)}$ = FALSE
    - if $F_d > F_{fk}$ then wheel $slip_{(i)}$ = TRUE

If for a time step, *wheel slip* = TRUE, then eq 18 needs to be used to solve *dv/dt*. eqs 19 & 20 should be used to calculate torque.

If for a time step, *wheel slip* = FALSE, then eq 14 needs to be used to solve *dv/dt*. eqs 12 & 5 should be used to calculate torque.

Analytical Solution to eq 14

While the analytical solution to eq 14 is not suitable to model robot acceleration (because it cannot account for wheel slippage) it is useful in checking the accuracy of the numerical solution and in setting an appropriate time step.

Staring with eq 14

$$\frac{dv}{dt} = \frac{Gn\tau_{MS} g_c}{mr_W}\left(1 - \frac{\tau_{ML}}{\tau_{MS}} - \frac{G}{2\pi r_W v_{MU}} v\right) \quad \text{eq 14}$$

rearranging

$$\frac{mr_W}{Gn\tau_{MS}g_c}\frac{1}{\left(1-\frac{\tau_{ML}}{\tau_{MS}}-\frac{G}{2\pi r_W v_{MU}}v\right)}dv = dt \qquad \text{eq 23}$$

This can be integrated

$$\frac{mr_W}{Gn\tau_{MS}g_c}\int\frac{dv}{\left(\frac{G}{2\pi r_W v_{MU}}v+\frac{\tau_{ML}}{\tau_{MS}}-1\right)}+\int dt = C_1 \qquad \text{eq 24}$$

or

$$\frac{2\pi r_W^2 m v_{MU}}{G^2 n\tau_{MS}g_c}\ln\left(1-\frac{\tau_{ML}}{\tau_{MS}}-\frac{G}{2\pi r_W v_{MU}}v\right)+t = C_1 \qquad \text{eq 25}$$

Solving for C_1 using the initial conditions @t = 0; v = 0

$$C_1 = \frac{2\pi r_W^2 m v_{MU}}{G^2 n\tau_{MS}g_c}\ln\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right) \qquad \text{eq 26}$$

The resulting equation becomes

$$\ln\left(1-\frac{\tau_{ML}}{\tau_{MS}}-\frac{G}{2\pi r_W v_{MU}}v\right) = \ln\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right)-\frac{G^2 n\tau_{MS}g_c}{2\pi r_W^2 m v_{MU}}t \qquad \text{eq 27}$$

Solving for v

$$v = \frac{2\pi r_W v_{MU}}{G}\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right)\left(1-e^{-\left(\frac{G^2 n\tau_{MS}g_c}{2\pi r_W^2 m v_{MU}}\right)t}\right) = \frac{dx}{dt} \qquad \text{eq 28}$$

Integrating a 2nd time to solve for x

$$\int dx+\frac{2\pi r_W v_{MU}}{G}\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right)\int e^{-\left(\frac{G^2 n\tau_{MS}g_c}{2\pi r_W^2 m v_{MU}}\right)t}dt-\frac{2\pi r_W v_{MU}}{G}\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right)\int dt = C_2 \qquad \text{eq 29}$$

$$x-\frac{4\pi^2 r_W^3 m v_{MU}^2}{G^3 n\tau_{MS}g_c}\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right)e^{-\left(\frac{G^2 n\tau_{MS}g_c}{2\pi r_W^2 m v_{MU}}\right)t}-\frac{2\pi r_W v_{MU}}{G}\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right)t = C_2 \qquad \text{eq 30}$$

Solving for C_2 using the initial conditions @t = 0; x = 0

$$C_2 = -\frac{4\pi^2 r_W^3 m v_{MU}^2}{G^3 n \tau_{MS} g_c}\left(1-\frac{\tau_{ML}}{\tau_{MS}}\right) \qquad \text{eq 31}$$

Yielding the analytical solution for x (position, ft)

$$x = \left(1-\frac{\tau_{ML}}{\tau_{MS}}\right)\left[\frac{2\pi r_W v_{MU}}{G}t - \frac{4\pi^2 r_W^3 m v_{MU}^2}{G^3 n \tau_{MS} g_c}\left(1-e^{-\left(\frac{G^2 n \tau_{MS} g_c}{2\pi r_W^2 m v_{MU}}\right)t}\right)\right] \qquad \text{eq 32}$$

By setting *wheel slip* manually and universally to FALSE, eqs 28 and 32 may be used to check the accuracy of the numerical method.

The Worksheet